Group Theory 5

01 October 2023 08:53

Homomorphism :-

Let $(G,*)$ and $(H,\circ)$ be groups Then a fuction $f: G \to H$ is a homomorphism if $\forall\, a,b \in G$,

$$\underset{\in H}{f(a*b)} = \underset{\in H}{f(a)\circ f(b)}$$

Isomorphism :- It is a homomorphism which is a bijection

$\hookrightarrow G \cong H$

$$f(ab) = f(a) f(b)$$

$$f(a_1 a_2 a_3 \cdot\, a_n) = f(a_1) f(a_2) \cdots f(a_n)$$

Theorem :- Let $f: (G,*) \longrightarrow (H,\circ)$ be a homomorphism

(i) $f(e) = e'$, where e' is the identity of H

(ii) If $a \in G$ then $f(a^{-1}) = (f(a))^{-1}$

(iii) If $a \in G$ and $n \in \mathbb{Z}$ then $f(a^n) = (f(a))^n$

$f(ab^{-1}) = f(a) f(b)^{-1} \to$ to prove homomorphism

Q) $f: X \to Y$ be a bijection and X, Y are sets. Show that $\alpha \mapsto f\circ\alpha\circ f^{-1}$ is an isomorphism on $S_X \to S_Y$.

Ans :- Suppose, $f\circ\alpha_1\circ f^{-1} = f\circ\alpha_2\circ f^{-1} \Rightarrow f^{-1}\circ f\circ\alpha_1\circ f^{-1} = f^{-1}\circ f\circ\alpha_2\circ f^{-1}$

$\Rightarrow \alpha_1\circ f^{-1} = \alpha_2\circ f^{-1} \Rightarrow \alpha_1 = \alpha_2$

One-One

$|X| = |Y|$

$\Rightarrow |S_X| = |S_Y|$ and one-one $\Rightarrow$ onto

$$\phi(\alpha_1 \alpha_2) = f \circ \alpha_1 \circ \alpha_2 \circ f^{-1} = f \circ \alpha_1 \circ f^{-1} \circ f \circ \alpha_2^{-1} \circ f^{-1}$$
$$= \phi(\alpha_1) \circ \phi(\alpha_2)$$

Q) Let G be a group and X be a set and $f : G \to X$ be a bijection Show that there is a unique operation on X so that X is a group and f is an isomorphism

Ans:- $|G| = |X|$

$G = \{a_1, a_2, \dots\}$ $\quad X = \{x_1, x_2, \dots \}$

$f(a_1 * a_2) = x_1 \circ x_2$ $\quad a_i * a_j \equiv x_i \circ x_j$ $\rightarrow$ corresponding sign here

$f(a_i) = x_i$ $\quad \forall i, j,\ f(a_i * a_j) = f(a_k) = x_k = x_i \circ x_j$

$= f(a_i) \circ f(a_j)$

Suppose another operation exists, let it be $\bullet$

for some i, j we must have $f(a_i * a_j) \neq x_i \bullet x_j$

$f(a_i) = x_i$ $\quad f(a_i * a_j) \neq f(a_i) \bullet f(a_j)$

So not homomorphism

Q) For any elements a, b in a group and any integer n, prove that $(a^{-1} b a)^n = a^{-1} b^n a$

Ans:- $(a^{-1} b a)^n = a^{-1} b a a^{-1} b a \dots a^{-1} b a = a^{-1} b^n a$

Q) Prove that if $(ab)^2 = a^2 b^2$ then $ab = ba$, $a, b \in G$

Ans:- $(ab)^2 = abab = aabb$

ab

Ans:- $(ab)^2 = abab = aabb$

$ba = ab$

Q) Let G be a group of exactly 4 elements. Prove that G is abelian.

Ans:- e, a, b, c $\quad ea = a = ae \quad eb = b = be$

$ec = c = ce$

WLOG, $c = ab$ or ba $\Rightarrow$ all pairs commute

but $ab, ba \in G \Rightarrow ab = ba$ $\Rightarrow$ Abelian

$\lbrace$ and G has 4 elements only

$ab = a$

$ba = \cancel{c}$ as $b = e$ then $\Rightarrow \Leftarrow$

Multiplicative Table:-

G	a_1	a_2		a_n
a_1	$a_1 * a_1$	$a_2 * a_1$		$a_n * a_1$
a_2	$a_1 * a_2$	$a_2 * a_2$		
	$\vdots$			
a_n	$a_1 * a_n$	$a_2 * a_n$		$a_n * a_n$

$\Rightarrow$ This has all the elements of G

Q) If G is a multiplicative group of all positive real numbers, show that $\ln : G \to (\mathbb{R}, +)$ is an isomorphism.

Ans:- $a \to \ln(a)$ $\Rightarrow$ one-one

$(\ln)^{-1}(b) \in G$ $\Rightarrow$ onto $\Rightarrow$ bijection

$\Rightarrow$ isomorphism

$\ln(ab) = \ln(a) + \ln(b)$ $\Rightarrow$ homomorphism

Q) Prove that a group G is abelian iff the function $f : G \to G$ defined by $f(a) = a^{-1}$ is a homomorphism.

Ans:- Suppose G is abelian,

$\forall a, b \in G$

Ans:— Suppose G is abelian,

$$ab = ba \quad \forall\, a, b \in G$$

$$f(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = f(a)f(b)$$

Suppose, $f(a) = a^{-1}$ is homomorphism,

$$f(ab) = f(a)f(b) \Rightarrow (ab)^{-1} = a^{-1}b^{-1} \Rightarrow (ab)^{-1} = (ba)^{-1}$$

$$\Rightarrow ab = ba$$